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Quantum gravidynamics III. Error estimates and perturbation expansions

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Abstract. The error of a kernel is expressed as an integral equation involving its error generator, which is obtained from the appropriate wave equation. An iterative solution of the integral equation leads to a perturbation expansion.

The relative errors of the covariant kernels derived in parts I and II are shown to be of second order in the interval, which is necessary if the kernels are to be used in path integrals.

Expressions for the error generators are derived. They involve the covariant derivatives of the world function and the spinor parallel propagator, exact expressions for which are given in an appendix.

1. Introduction

Numerical analysts like to say that an approximation is worthless without an error estimate. Perhaps this is an extreme view, but it does contain an important truth. We shall not be confident in the covariant kernels derived in parts I and II (Clutton-Brock 1975a, b) until we have justified them by a proper error estimate. In order to use an approximate kernel in a path integral, the relative error of the kernel must be of the second order in the interval or smaller. We verify in this part III that the relative error of our covariant kernels is indeed of second order.

An approximate kernel $\hat{K}$ does not obey the appropriate wave equation $\hat{K}\overleftarrow{L} = \mathscr{I}$: there is a residual

$$\mathscr{E} = \hat{K}\overleftarrow{L} - \mathscr{I} \tag{1.1}$$

which is the error generator. We shall see in § 2 that the error of a kernel is given in terms of its error generator by a 4-content integral

$$\hat{K} - K = \int \mathscr{E} K \, \mathrm{d}C \tag{1.2}$$

which is an integral equation for the exact kernel K. An obvious interation yields a perturbation expansion.

We obtain in § 3 an order of magnitude estimate of the errors from a simple approximation to the error generators in a special coordinate system. More general and more exact expressions for the error generators, suitable for use in a perturbation expansion, are derived in § 4.

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2. Perturbation expansions based on the error generator

The kernel expresses the wavefunction at a point x'' as an integral over a 3-surface V' enclosing x''. By means of Gauss's theorem we can transform the 3-surface integral into an integral over the 4-content enclosed by V'. For a Klein–Gordon particle we obtain

$$\hat{\psi}(x'') = \int_{C'} [\hat{K}_{(\mathrm{KG})}(x'', x)(-\overleftarrow{\partial}^k\overleftarrow{\mathrm{D}}_k + m^2)]\psi(x)\,\mathrm{d}C, \tag{2.1}$$

and for a Dirac particle we obtain

$$\hat{\psi}(x'') = \int_{C} [\hat{K}_{(\mathrm{D})}(x'', x)(\overleftarrow{\mathrm{D}}_k\gamma^k - \mathrm{i}m)]\psi(x)\,\mathrm{d}C, \tag{2.2}$$

where $\hat{\psi}$ is the approximate wavefunction generated by the approximate kernel $\hat{K}$. If the kernel were exact, the quantity in square brackets would be the identity kernel $\mathscr{I}$. The departure $\mathscr{E}$ of the square brackets from the identity kernel generates an error in the wavefunction, so I call $\mathscr{E}$ the *error generator*. The error generator of the Klein–Gordon kernel is

$$\mathscr{E}_{(\mathrm{KG})}(x'', x') = \hat{K}_{(\mathrm{KG})}(x'', x')(-\overleftarrow{\partial}^{k'}\overleftarrow{\mathrm{D}}'_k + m^2) - \mathscr{I}(x'', x'), \tag{2.3}$$

and the error generator of the Dirac kernel is

$$\mathscr{E}_{(\mathrm{D})}(x'', x') = \hat{K}_{(\mathrm{D})}(x'', x')(\overleftarrow{\mathrm{D}}'_k\gamma^k - \mathrm{i}m) - \mathscr{I}(x'', x'). \tag{2.4}$$

The error in the wavefunction is

$$\hat{\psi}(x'') - \psi(x'') = \int_{C'} \mathscr{E}(x'', x)\psi(x)\,\mathrm{d}C, \tag{2.5}$$

and since $\psi(x)$ can be expressed in terms of the kernel, there is a similar relation for the kernel:

$$\hat{K}(x'', x') - K(x'', x') = \int_{C'} \mathscr{E}(x'', x)K(x, x')\,\mathrm{d}C. \tag{2.6}$$

This is an integral equation for the exact kernel K which, when solved by iteration, yields the perturbation expansion

$$K(x'', x') = \hat{K}(x'', x') - \int_{C'} \mathscr{E}(x'', x_1)\hat{K}(x_1, x')\,\mathrm{d}C_1 + \int\int \mathscr{E}(x'', x_1)\mathscr{E}(x_1, x_2)\hat{K}(x_2, x')\,\mathrm{d}C_1\,\mathrm{d}C_2 + \dots . \tag{2.7}$$

The perturbation expansions used in flat space–time generally cover the whole of space–time, but in curved space–time this will not always be possible. For in a large region initially diverging geodesics may converge and eventually cross. For two points at which geodesics cross, the world function and parallel propagators will not be uniquely defined, and in the neighbourhood of the two points the partial derivatives of the world function and parallel propagators will be singular. We shall see that the error generator depends on the covariant derivatives of the world function and the spinor parallel propagator, so we may expect that perturbation expansions covering such a large region will be divergent. It will therefore be necessary to restrict the expansion to a region C' which is small enough that the geodesics do not cross.

It may be possible to define a physical process of interest by a transition probability between two wave packets covering a sufficiently small region of space–time, in which case the process can be analysed by a single perturbation expansion. If this is not possible, it will be necessary to calculate the transition amplitude as a path integral. The perturbation expansion will still be important for the direct calculation of a path integral by the Monte Carlo method, however, because we can thereby increase the size of the steps for a given error bound and so reduce the variance.

3. The order of magnitude of the error

A path integral with intervals $x'' \leftarrow x'$ of order Δ will have $N = O(\Delta^{-1})$ steps. If the approximate kernels have relative errors of ϵ, so that

$$\hat{K} = (1+\epsilon)K \tag{3.1}$$

then the relative error of the path integral is

$$(1+\epsilon)^N - 1 = O(\Delta^{-1}\epsilon). \tag{3.2}$$

So to ensure that a path integral has a first-order error, we need to ensure that the kernel has a second-order error. We now verify that the covariant kernels derived in parts I and II do indeed have a second-order error.

A referee suggested an ingenious method of obtaining a simple approximation to the error generator, which is adequate for our present purposes and more revealing than an exact expression. We choose a coordinate system and a tetrad field for which the kernel is formally identical with the flat-space kernel. Then the error generator is obtained from the extra terms in the covariant wave equations.

We use a quasi-cartesian coordinate system with coordinates

$$X^A(x) = -W^A(x'', x) = (u-u'')\left(\frac{\mathrm{d}x^k}{\mathrm{d}u}\right)_{u''} \lambda^A_k(x''), \tag{3.3}$$

where $W^A(x'', x)$ is the covariant derivative of the world function. The metric is given by Synge (1960) as

$$g_{AB}(x) = \eta_{AB} + \tfrac{1}{3}R_{AMNB}X^M X^N + O(\Delta^3). \tag{3.4}$$

The tensor connection is

$$\Gamma^C_{AB} = -\tfrac{1}{3}(R^C_{.ABM} + R^C_{.BAM})X^M + O(\Delta^2), \tag{3.5}$$

so the Klein–Gordon operator is

$$\begin{aligned}(-\mathrm{D}_A\partial^A + m^2) &= [-g^{AB}(\partial_A\partial_B - \Gamma^C_{AB}\partial_C) + m^2] \\ &\simeq (-\eta^{AB}\partial_A\partial_B + m^2 - \tfrac{2}{3}R^B_A X^A\partial_B - \tfrac{1}{3}R^{A..B}_{.MN.}X^M X^N\partial_A\partial_B).\end{aligned} \tag{3.6}$$

In these coordinates the Klein–Gordon kernel

$$\hat{K}_{(\mathrm{KG})}(x'', x') = \int \exp[\mathrm{i}q_A W^A(x'', x')]\,\mathrm{d}\Omega(q) = \int \exp(-\mathrm{i}q_A X^A)\,\mathrm{d}\Omega(q) \tag{3.7}$$

obeys the flat-space wave equation

$$(-\eta^{AB}\partial_A\partial_B+m^2)\hat{K}_{(KG)} = \delta^{(4)}(X) = \mathscr{I}(x'', x'), \tag{3.8}$$

and so the error generator is

$$\mathscr{E}_{(KG)} \simeq -(\tfrac{2}{3}R^B_A X^A\partial_B+\tfrac{1}{3}R^{A\,..\,B}_{\,.MN.}X^M X^N\partial_A\partial_B)\hat{K}_{(KG)}. \tag{3.9}$$

The tetrad frame which makes the spinor parallel propagator formally identical to the identity matrix I is obtained by parallel propagation from x'':

$$\lambda^A_\alpha(x) = \eta^A_\alpha - \int_0^u \Gamma^A_{N\alpha}\frac{dX^N}{du}\,du = \eta^A_\alpha+\tfrac{1}{6}R^A_{.M\alpha N}X^M X^N+O(\Delta^3). \tag{3.10}$$

The spinor connection is

$$\Gamma_A = -\tfrac{1}{4}\sigma^{\mu\nu}\lambda_{B\mu}\lambda^B_{\nu|A} = \tfrac{1}{24}\sigma^{\mu\nu}(2R_{\mu\nu AM}+R_{\mu A\nu M}-R_{\nu A\mu M})X^M+O(\Delta^2), \tag{3.11}$$

and so the Dirac operator is

$$\begin{aligned}(\overleftarrow{D}_A\gamma^A-im) &= [(\overleftarrow{\partial}_A-\Gamma_A)\lambda^A_\alpha\gamma^\alpha-im]\\ &\simeq [(\overleftarrow{\partial}_A\eta^A_\alpha\gamma^\alpha-im)+\tfrac{1}{6}\overleftarrow{\partial}_A R^A_{.M\alpha N}X^M X^N\gamma^\alpha\\ &\quad-\tfrac{1}{24}\sigma^{\mu\nu}(2R_{\mu\nu\alpha M}+R_{\mu\alpha\nu M}-R_{\nu\alpha\mu M})\gamma^\alpha X^M].\end{aligned} \tag{3.12}$$

In these coordinates and in this tetrad frame the Dirac kernel obeys the flat-space wave equation

$$\hat{K}_{(D)}(\overleftarrow{\partial}_A\eta^A_\alpha\gamma^\alpha-im) = \delta^{(4)}(X)I = \mathscr{I}(x'', x'), \tag{3.13}$$

and so the error generator is

$$\mathscr{E}_{(D)} \simeq \hat{K}_D\{\tfrac{1}{6}\overleftarrow{\partial}_A R^A_{.M\alpha N}X^M X^N-\tfrac{1}{24}\sigma^{\mu\nu}(2R_{\mu\nu\alpha M}+R_{\mu\alpha\nu M}-R_{\nu\alpha\mu M})X^M\}\gamma^\alpha. \tag{3.14}$$

The Klein–Gordon kernel connects wavefunctions through the relationship

$$\int \hat{K}_{(KG)}(\vec{\partial}'-\overleftarrow{\partial}')\psi'\,dV' = \psi'' = O(1), \tag{3.15}$$

and since

$$\int dV = O(\Delta^3), \tag{3.16}$$

the Klein–Gordon kernel must be of order

$$\hat{K}_{(KG)} = O(\Delta^{-2}), \qquad \partial\hat{K}_{(KG)} = O(\Delta^{-3}), \qquad \partial\partial\hat{K}_{(KG)} = O(\Delta^{-4}). \tag{3.17}$$

Similarly, the Dirac kernel must be of order

$$\hat{K}_{(D)} = O(\Delta^{-3}), \qquad \partial\hat{K}_{(D)} = O(\Delta^{-4}). \tag{3.18}$$

By combining (3.17) with (3.9) we find

$$\mathscr{E}_{(KG)} = O(\Delta^{-2}), \tag{3.19}$$

and by combining (3.18) with (3.14) we find

$$\mathscr{E}_{(D)} = O(\Delta^{-2}). \tag{3.20}$$

The relative error in the kernel is

$$\epsilon = K^{-1}(\hat{K}-K) = K^{-1}\int \mathscr{E}K\,\mathrm{d}C, \tag{3.21}$$

and since

$$\int \mathrm{d}C = \mathrm{O}(\Delta^4), \qquad \mathscr{E} = \mathrm{O}(\Delta^{-2}), \tag{3.22}$$

we find that

$$\epsilon = \mathrm{O}(\Delta^2), \tag{3.23}$$

so that the relative error of the kernel is indeed of the second order in the interval.

4. Evaluation of the error generator

To use the error generator in a perturbation expansion, we need an expression which does not put x'' at the origin of its coordinate system. So we now evaluate the error generator in an arbitrary coordinate system.

The flat-space Klein–Gordon kernel is a scalar function $k(\dots)$ of ξ where

$$\xi = \tfrac{1}{2}\eta_{ab}(x''-x')^a(x''-x')^b. \tag{4.1}$$

The covariant Klein–Gordon kernel is the same function $k(\dots)$ of the world function $W = W(x'', x')$ which is defined by Synge (1960) as half the square of the geodesic measure between x'' and x'. Hence the error generator is

$$\begin{aligned}\mathscr{E}_{(\mathrm{KG})}(x'', x') &= (-\mathrm{D}'_a\partial^{a'}+m^2)k(W)-\mathscr{I}(x'', x')\\ &= -W(x'', x')^a W(x'', x')_a\partial_W^2 k(W) - W(x'', x')^a_a\partial_W k(W)\\ &\quad + m^2 k(W) - \mathscr{I}(x'', x').\end{aligned} \tag{4.2}$$

Making use of the coincidence limit

$$\lim_{x''\to x'} W(x'', x')^a_a = 4, \tag{4.3}$$

and of the relation

$$W(x'', x')^a W(x'', x')_a = 2W(x'', x'), \tag{4.4}$$

we rewrite (4.2) as

$$\mathscr{E}_{(\mathrm{KG})}(x'', x') = [4-W(x'', x')^a_a]\partial_W k(W) + [(-2W\partial_W^2 - 4\partial_W + m^2)k(W) - \mathscr{I}(x'', x')]. \tag{4.5}$$

In flat space–time, the kernel is exact, the error generator vanishes, and so

$$(-2W\partial_W^2 - 4\partial_W + m^2)k(W) = \mathscr{I}(x'', x'). \tag{4.6}$$

Except at the point $x'' = x'$ this is just a property of the function $k(\dots)$, and since in a small neighbourhood of $x'' = x'$ the effects of curvature may be neglected, (4.6) must be true in a curved space–time as well. Hence the error generator of the Klein–Gordon kernel is

$$\mathscr{E}_{(\mathrm{KG})}(x'', x') = [4-W(x'', x')^a_a]\partial_W k(W). \tag{4.7}$$

The Dirac kernel with x_q at x'' is

$$\hat{K}_{(D)}(x'', x') = \mathrm{i}\int \exp[\mathrm{i}q_\alpha W^\alpha(x'', x')]w(q)\,\mathrm{d}\Omega(q)\Lambda(x'', x'). \tag{4.8}$$

If we multiply the integrand by

$$-\mathrm{i}w_+(q) = -\mathrm{i}(\gamma^\beta q_\beta + m) \tag{4.9}$$

where $w_+(q)$ is such that

$$w_+(q)w(q) = w(q)w_+(q) = \Omega(q) \tag{4.10}$$

$$w(q)w_+(q)\,\mathrm{d}\Omega(q) = \frac{1}{(2\pi)^4}\,\mathrm{d}^4q, \tag{4.11}$$

then we get

$$\int \exp[\mathrm{i}q_\alpha W^\alpha(x'', x')]w(q)w_+(q)\,\mathrm{d}\Omega(q)\Lambda(x'', x') = \mathscr{I}(x'', x'). \tag{4.12}$$

The error generator of the Dirac kernel is

$$\begin{aligned}\mathscr{E}_{(D)}(x'', x') &= \hat{K}_{(D)}(x', x')(\overleftarrow{\mathrm{D}}'_\mu\gamma^\mu - \mathrm{i}m) - \mathscr{I}(x'', x')\\ &= \mathrm{i}\int \exp[\mathrm{i}q_\alpha W^\alpha(x'', x')]w(q)\,\mathrm{d}\Omega(q)\\ &\qquad \times\{\Lambda(x'', x')_\mu\gamma^\mu + \Lambda(x'', x')[\mathrm{i}q_\alpha W^\alpha(x'', x')_\mu\gamma^\mu - \mathrm{i}m]\} - \mathscr{I}(x'', x').\end{aligned} \tag{4.13}$$

Using the commutation relation

$$\Lambda(x'', x')\gamma^\mu = \gamma^\beta g_\beta(x'', x')^\mu\Lambda(x'', x'), \tag{4.14}$$

we find

$$\begin{aligned}&\Lambda(x'', x')[\mathrm{i}q_\alpha W^\alpha(x'', x')_\mu\gamma^\mu - \mathrm{i}m]\\ &\quad = -\mathrm{i}w_+(q)\Lambda(x'', x') + \mathrm{i}q_\alpha\gamma^\beta[W^\alpha(x'', x')_\mu g^\mu(x', x'')_\beta + \eta^\alpha_\beta]\Lambda(x'', x').\end{aligned} \tag{4.15}$$

The error generator of the Dirac kernel is

$$\begin{aligned}\mathscr{E}_{(D)}(x'', x') &= \int \exp[\mathrm{i}q_\alpha W^\alpha(x'', x')]w(q)\,\mathrm{d}\Omega(q)\\ &\qquad \times\{\mathrm{i}\Lambda(x'', x')_\mu\gamma^\mu - q_\alpha\gamma^\beta[W^\alpha(x'', x')_\mu g^\mu(x', x'')_\beta + \eta^\alpha_\beta]\Lambda(x'', x')\}.\end{aligned} \tag{4.16}$$

Synge (1960) gives approximate expressions for the covariant derivatives of the world function, from which we have

$$4 - W(x'', x')^\alpha_\alpha = -\int_{u'}^{u''}\frac{(u''-u)^2}{u''-u'}R_{kl}\frac{\mathrm{d}x^k}{\mathrm{d}u}\frac{\mathrm{d}x^l}{\mathrm{d}u}\,\mathrm{d}u + \mathrm{O}(\Delta^4), \tag{4.17}$$

$$\begin{aligned}&W^\alpha(x'', x')_\mu g^\mu(x', x'')_\beta + \eta^\alpha_\beta\\ &\quad = \int_{u'}^{u''}\frac{(u''-u)(u-u')}{u''-u'}g^\alpha(x'', x)^a g_\beta(x'', x)^b R_{aklb}\frac{\mathrm{d}x^k}{\mathrm{d}u}\frac{\mathrm{d}x^l}{\mathrm{d}u}\,\mathrm{d}u + \mathrm{O}(\Delta^4).\end{aligned} \tag{4.18}$$

If we apply Synge's method for the covariant derivatives of the tensor parallel propagator to the spinor parallel propagator, we obtain

$$\Lambda(x'', x')_\mu = \int_{u'}^{u''} \Lambda(x'', x)\Gamma_{[k|l]}\Lambda(x, x')\left(\frac{u''-u}{u''-u'}\right) g^k(x, x')_\mu \frac{\mathrm{d}x^l}{\mathrm{d}u}\,\mathrm{d}u + O(\Delta^3). \tag{4.19}$$

Here $\Gamma_{[k|l]}$ is the commutator of the covariant derivative of the spinor parallel propagator. These expressions are adequate for first-order perturbation expansions. For higher-order perturbation expansions we need more accurate expressions, which we derive in the appendix.

5. General conclusions

It is now time to take stock of our exploration of quantum gravidynamics. What has been achieved, what are the major limitations of the path integral approach, and what should be done to improve it?

The covariant kernels have second-order errors, and so are equivalent to a flat-space kernel with first-order perturbations. The analogous kernel in quantum electrodynamics is

$$\hat{K}_\phi(x'', x') = \exp\left(\mathrm{i}\int_{x'}^{x''} \phi_k(x)\,\mathrm{d}x^k\right) K_F(x'', x'), \tag{5.1}$$

where K_F is the free-particle kernel. Thus first-order effects can be predicted by the covariant kernels directly, and second-order effects by first-order perturbation expansions.

Those physical processes that can be defined over a restricted region of space–time can be analysed using the covariant kernels and perturbation expansions based on them. However, in the Compton era, when the radius of curvature of the universe is less than the Compton wavelength of typical particles, physical processes will cover regions containing many radii of curvature. Then geodesics may well cross in the region of interest, and perturbation expansions cannot cover the entire region. There is then no alternative that I can see to a path integral, with steps small enough so that the geodesic over each step is unique.

The techniques for evaluating path integrals are as yet undeveloped. The crude Monte Carlo method outlined in part I would of course be hopless. However, it may be possible to develop sufficiently sophisticated methods of variance reduction to turn a refined Monte Carlo method into a workable method. The major problem is the large amount of phase cancellation that occurs on paths far from the classical path of stationary action. It should be possible to overcome this by using a variant of the 'correlated sampling' technique, in which single paths are replaced by bundles of paths. The individual paths in each bundle may be weighted so that paths near the classical path are given more weight than paths far from the classical path ('importance sampling'). The information gained during the process of generating the paths may be used to multiply the paths in some bundles and to cancel paths in other bundles ('Russian roulette and splitting').

It would be even more useful if we could develop analytic approximations to path integrals, perhaps in spaces of particular symmetry such as the Kerr or Robertson–Walker metrics.

In part I we referred to the problem of interpreting wavefunctions: this is an important need for *any* theory of quantum gravidynamics. The interpretation should be based on the analysis of conceivable measurements. In a space of small curvature it should be possible to put the whole measurement inside a region in which geodesics do not cross: then the measurement may be analysed using perturbation techniques. For spaces of large curvature, we must again turn to path integrals.

In spite of the technical difficulties, If feel that the path integral approach provides a well defined and conceptually simple framework for quantum gravidynamics.

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Appendix. Exact evaluation of the covariant derivatives of the world function and the spinor parallel propagator

Consider a family of geodesics, starting from a fixed point x'', and ending on a curve $x' = x'(v)$. Along the geodesics there is an affine parameter u which has the fixed end values u'' and u' at x'' and x', so the geodesics span a 2-space $x(u, v)$. In this 2-space we define the vectors

$$U^k(x) = \frac{\partial x^k}{\partial u}, \qquad V^k(x) = \frac{\partial x^k}{\partial v}. \tag{A.1}$$

Now $V^k(x)$ at any point x on a geodesic depends linearly on its value $V^{k'}$ at the end point x'. There must be a tensor $\mathscr{D}^k(x, x'', x')_m$ connecting V at x with V' at x':

$$V^k(x) = \left(\frac{u''-u}{u''-u'}\right)\mathscr{D}^k(x, x'', x')_m V^{m'}. \tag{A.2}$$

We call this tensor the *deviation tensor*. From the differential equation of geodesic deviation, Synge (1960) derives an integral equation for the behaviour of V along the geodesic. The deviation tensor obeys substantially the same integral equation:

$$\mathscr{D}^k(x, x'', x')_m = g^k(x, x')_m + \left(\int_u^{u''} \frac{(u''-\bar{u})(u-u')}{u''-u} + \int_{u'}^{u} (\bar{u}-u')\right) d\bar{u}$$
$$\times\, g^k(x, \bar{x})_a \bar{R}^a{}_{.pqb} \bar{U}^p \bar{U}^q \left(\frac{u''-\bar{u}}{u''-u'}\right)\mathscr{D}^b(\bar{x}, x'', x')_m, \tag{A.3}$$

which is readily solved by iteration to any desired degree of accuracy. The deviation tensor represents a fundamental property of space–time, and the difference between the deviation tensor and the parallel propagator is the global analogue of the curvature tensor.

The arguments leading to the covariant derivatives of the world function follow closely those of Synge, except that we use the deviation tensor where he uses its first approximation, the parallel propagator. Since

$$W^k(x'', x') = (u''-u')U^k(x''), \qquad W(x'', x')^k = -(u''-u')U^k(x'), \tag{A.4}$$

we have

$$W^k(x'', x')_m V^{m'} = (u''-u')\mathrm{D}_v U^k(x'') = (u''-u')\mathrm{D}_u V^k(x''), \tag{A.5}$$

$$W(x'', x')^k_m V^{m'} = -(u''-u')\mathrm{D}_u V^k(x'). \tag{A.6}$$

Now we use (A.2) and (A.3) to differentiate V with respect to u:

$$(u''-u')\mathrm{D}_u V^k(x)$$

$$= -g^k(x, x')_m V^{m'} + \left(\int_u^{u''} (u''-\bar{u}) - \int_{u'}^{u} (\bar{u}-u') \right) \mathrm{d}\bar{u}$$

$$\times g^k(x, \bar{x})_a \bar{R}^a{}_{.pqb} \bar{U}^p \bar{U}^q \left(\frac{u''-\bar{u}}{u''-u'} \right) \mathscr{D}^b(\bar{x}, x'', x')_m V^{m'}. \tag{A.7}$$

On substituting into (A.5) and (A.6), and cancelling the arbitrary $V^{m'}$, we find

$$W^k(x'', x')_m = -g^k(x'', x')_m + \int_{u'}^{u''} \frac{(u''-u)(u-u')}{u''-u'} g^k(x'', x)_a R^a{}_{.pqb} U^p U^q \mathscr{D}^b(x, x'', x')_m \, \mathrm{d}u, \tag{A.8}$$

$$W(x'', x')^k_m = g^k_m + \int_{u'}^{u''} \frac{(u''-u)^2}{u''-u'} g^k(x', x)_a R^a{}_{.pqb} U^p U^q \mathscr{D}^b(x, x'', x')_m \, \mathrm{d}u. \tag{A.9}$$

To find the covariant derivative of the spinor parallel propagator, we consider two spinors $\bar{\psi}$ and ϕ which are carried by parallel transport along the geodesic:

$$\bar{\psi}(x) = \bar{\psi}(x'')\Lambda(x'', x), \qquad \phi(x) = \Lambda(x, x')\phi(x'). \tag{A.10}$$

We start with

$$\bar{\psi}(x)\overleftarrow{\mathrm{D}}_v = \bar{\psi}(x'')\Lambda(x'', x)_\mu V^\mu(x), \tag{A.11}$$

and since

$$\mathrm{D}_u \phi = 0, \qquad \bar{\psi}\overleftarrow{\mathrm{D}}_u = 0, \tag{A.12}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}u}(\bar{\psi}\overleftarrow{\mathrm{D}}_v \phi) = \bar{\psi}\overleftarrow{\mathrm{D}}_v \overleftarrow{\mathrm{D}}_u \phi = \bar{\psi}(\overleftarrow{\mathrm{D}}_v \overleftarrow{\mathrm{D}}_u - \overleftarrow{\mathrm{D}}_u \overleftarrow{\mathrm{D}}_v)\phi. \tag{A.13}$$

Recall that the covariant derivative of an adjoint spinor is

$$\bar{\psi}\overleftarrow{\mathrm{D}}_k = \bar{\psi}(\overleftarrow{\partial}_k - \Gamma_k) \qquad \text{with } \Gamma_k = -\tfrac{1}{4}\sigma^{\alpha\beta}\lambda_{a\alpha}\lambda^a_{\beta|k}. \tag{A.14}$$

It is easy to verify that the σ matrices commute, and so the spinor connection matrices Γ_k commute. Hence the commutator of the covariant derivatives of an adjoint spinor is

$$\bar{\psi}(\overleftarrow{\mathrm{D}}_v \overleftarrow{\mathrm{D}}_u - \overleftarrow{\mathrm{D}}_u \overleftarrow{\mathrm{D}}_v) = \bar{\psi}\overleftarrow{\mathrm{D}}_{[k}\overleftarrow{\mathrm{D}}_{l]} V^k U^l = -\bar{\psi}\Gamma_{[k|l]} V^k U^l, \tag{A.15}$$

where

$$\Gamma_{[k|l]} = -\tfrac{1}{4}\sigma^{\alpha\beta}\eta_{ab}(\lambda^a_{\alpha|[l}\lambda^b_{\beta|k]} + \lambda^a_\alpha \lambda^b_{\beta|[k|l]}). \tag{A.16}$$

Going back to (A.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}u}(\bar{\psi}\overleftarrow{\mathrm{D}}_v \phi) = -\bar{\psi}\Gamma_{[k|l]}\phi V^k U^l, \tag{A.17}$$

and so

$$(\bar{\psi}\overleftrightarrow{\mathrm{D}}_{v}\phi)_{x=x'} = \bar{\psi}(x'')\Lambda(x'', x')_{\mu}V^{\mu'}\phi(x') = -\int_{u'}^{u''}\frac{\mathrm{d}}{\mathrm{d}u}(\bar{\psi}\overleftrightarrow{\mathrm{D}}_{v}\phi)\,\mathrm{d}u$$

$$= \bar{\psi}(x'')\int_{u'}^{u''}\Lambda(x'', x)\Gamma_{[k|l]}\Lambda(x, x')V^{k}U^{l}\,\mathrm{d}u\phi(x'). \qquad \text{(A.18)}$$

Substituting for V^k from (A.2), and cancelling the arbitrary $\bar{\psi}''$, ϕ' and $V^{\mu'}$, we find

$$\Lambda(x'', x')_{\mu} = \int_{u'}^{u''}\left(\frac{u''-u}{u''-u'}\right)\Lambda(x'', x)\Gamma_{[k|l]}\Lambda(x, x')\mathscr{D}^{k}(x, x'', x')_{\mu}U^{l}\,\mathrm{d}u. \qquad \text{(A.19)}$$

The approximate expressions quoted at the end of § 4 may be obtained from (A.8), (A.9) and (A.19) by substituting the tensor parallel propagator for the deviation tensor.

References

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